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Ondřej Kreml

Šárka Nečasová

Tomasz Piasecki

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Ondřej Kreml [1]   Šárka Nečasová [1]   Tomasz Piasecki [2]

[1]Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha Czech Republic

[2] Institute of Applied Mathematics and Mechanics, University of Warsaw
Banacha 2, 02-097 Warszawa, Poland

Abstract

We prove the local well-posedness for the barotropic compressible Navier-Stokes system on a moving domain, a motion of which is determined by a given vector field $\mathbf{V}$, in a maximal $L_p - L_q$ regularity framework. Under additional smallness assumptions on the data we show that our solution exists globally in time and satisfies a decay estimate. In particular, for the global well-posedness we don't require exponential decay or smallness of $\mathbf{V}$ in $L_p(L_q)$. However, we require exponential decay and smallness of its derivatives.

Keywords: compressible Navier-Stokes equations, moving domain, strong solution, local existence, maximal regularity

MSC: 35Q30, 76N10

1 Introduction

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We consider a barotropic flow of a compressible viscous fluid in the absence of external forces described by the isentropic compressible Navier-Stokes system

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$ i1a

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$ i1b

where ϱ is the density of the fluid and $\mathbf{u}$ denotes the velocity. We assume that the stress tensor $\mathbb{S}$ is determined by the standard Newton rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \zeta \operatorname{div}_x \mathbf{u} \mathbb{I} \tag{1.3}$$ i4

with constant viscisity coefficients $\mu > 0$ and $\zeta \geq 0$. The pressure $p(\varrho)$ is a given sufficiently smooth function of the density. We assume the fluid occupies a time-dependent bounded domain Ω_t, the motion of which is described by means of a given velocity field $\mathbf{V}(t,x)$, where $t \geq 0$ and $x \in \mathbb{R}^3$. More precisely, we assume that if $\mathbf{X}$ solves the following system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t,x) = \mathbf{V}\Big(t, \mathbf{X}(t,x)\Big), \ t > 0, \ \mathbf{X}(0,x) = x,$$

we set

$$\Omega_\tau = \mathbf{X}\left(\tau, \Omega_0\right),$$

where $\Omega_0 \subset \mathbb{R}^3$ is a given bounded domain at initial time $t = 0$. Moreover we denote $\Gamma_\tau = \partial\Omega_\tau$ and

$$Q_\tau = \bigcup_{t \in (0,\tau)} \{t\} \times \Omega_t =: (0,\tau) \times \Omega_t.$$

We consider system (1.1)-(1.2) supplied with the Dirichlet boundary conditions

$$(\mathbf{u} - \mathbf{V})|_{\Gamma_\tau} = 0 \text{ for any } \tau \geq 0 \tag{1.4}$$ i6a

and the initial conditions

$$\varrho(0,\cdot) = \varrho_0, \quad \mathbf{u}(0,\cdot) = \mathbf{u}_0 \quad \text{in } \Omega_0. \tag{1.5}$$ i7

The existence theory for system (1.1)-(1.2) on fixed domains is nowadays quite well developed. The existence of global weak solutions has been first established by Lions [16]. This result has been later extended by Feireisl and coauthors ([10], [4], [5], [6]) to cover larger class of pressure laws. Strong solutions on fixed domains are known to exists locally in time or globally provided certain smallness assumptions on the data. For no-slip boundary conditions see among others [18], [19], [28], [29] for the results in Hilbert spaces, [20], [21], [22] in L_p setting and [3] for a maximal $L_p - L_q$ regularity approach. Problem with slip boundary conditions on a fixed domain has been investigated by Zajaczkowski [30], Hoff [12] and, more recently, by Shibata and Murata [17], [26] in the $L_p - L_q$ maximal regularity setting. In [23],[23] the approach from [3] has been adapted to treat a generalization of compressible Navier-Stokes system describing flow of a compressible mixture with cross-diffusion. For results on free boundary problems for system (1.1)-(1.2) we refer to [31], [32] where global existence of strong solutions in L_2-setting has been shown under the assumption that the domain is close to a ball and to [25] where a free boundary problem is treated in $L_p - L_q$ approach.

The existence theory for system (1.1)-(1.2) on a moving domain with given motion of the boundary started to develop with the results for weak solutions obtained using a penalization method in [8] for no-slip bondary conditions and [9] for slip conditions. These results have been recently generalized to the complete system with heat conductivity in [13] and [14]. The first weak-strong uniqueness result on a moving domain has been shown in [2] in case of no-slip boundary condition. A generalization of this to slip conditions as well as a local existence result for strong solution for both types boundary conditions can be found in [15]. There, the authors use the energy approach in L_2 setting for the existence result.

The aim of this paper is to extend the existence theory for strong solutions on a moving domain to $L_p - L_q$ maximal regularity setting. We present a more detailed outline of the proof after stating our main result, however first let us resume the notation used in the paper.

1.1 Notation

We use standard notation for Lebesgue spaces $L_p(\Omega)$ and Sobolev spaces $W^k_p(\Omega)$ with $k \in \mathbb{N}$ on a fixed domain Ω. By $C_B(\Omega)$ we denote a space of bounded continuous functions on Ω. Furthermore, for a Banach space X, $L_p(0,T;X)$ is a Bochner space of functions for which the norm

$$\|f\|_{L_p(0,T;X)} = \begin{cases} \left(\int_0^T \|f(t)\|_X \, \mathrm{d}t\right)^{1/p}, & 1 \le p < \infty, \\ \text{ess sup}_{0\le t\le T}\|f(t)\|_X, & p = \infty \end{cases}$$

is finite. Then

$$W^1_p(0,T;X) = \{f \in L_p(0,T;X) : \ \partial_t f \in L_p(0,T;X)\}.$$

For $p \ge 1$ we denote by p' its dual exponent, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Next, we recall that for $0 < s < \infty$ and m a smallest integer larger than s we define Besov spaces on domains as intermediate spaces

$$B^s_{q,p}(\Omega) = (L_q(\Omega), W^m_q(\Omega))_{s/m,p}, \tag{1.6}$$

def:bsqp(

where $(\cdot,\cdot)_{s/m,p}$ is the real interpolation functor, see [1, Chapter 7]. In particular,

$$B^{2(1-1/p)}_{q,p}(\Omega) = (L_q(\Omega), W^2_q(\Omega))_{1-1/p,p} = (W^2_q(\Omega), L_q(\Omega))_{1/p,p}. \tag{1.7}$$

def:bsqp

We shall not distinguish between notation of spaces for scalar and vector valued functions, i.e. we write $L_q(\Omega)$ instead of $L_q(\Omega)^3$ etc. However, we write vector valued functions in boldface.

For function spaces on moving domains we assume that there exists $R > 0$ such that for all $t \in [0,T]$ it holds $\Omega_t \subset B_R(0)$, where $B_R(0)$ denotes the ball in $\mathbb{R}^3$ of radius R centered at the origin. Then we define

$$L_p(0,T;L_q(\Omega_t)) := \{u \in L_p(0,T;L_q(B_R(0))), u(t,\cdot) = 0 \text{ in } B_R(0)\setminus\Omega_t \text{ for a.e. } t \in (0,T)\} \tag{1.8}$$

eq:prost

with the norm

$$\|u\|_{L_p(0,T;L_q(\Omega_t))} := \left(\int_0^T \|u(t)\|^p_{L_q(\Omega_t)} \, \mathrm{d}t\right)^{\frac{1}{p}}$$

for $p < \infty$ and

$$\|u\|_{L_\infty(0,T;L_q(\Omega_t))} := \text{ess sup}_{t\in(0,T)} \|u(t)\|_{L_q(\Omega_t)}.$$

Similarly we define spaces $L_p(0,T;W^l_q(\Omega_t))$. Let $l \in \mathbb{N}$ and α be a multi-index. Then

$$L_p(0,T;W^l_q(\Omega_t)) := \{u \in L_p(0,T;L_q(\Omega_t)), \partial^\alpha u \in L_p(0,T;L_q(\Omega_t)) \ \ \forall\,|\alpha| \le l\}$$

with the norm

$$\|u\|_{L_p(0,T;W^l_q(\Omega_t))} := \sum_{|\alpha|\le l} \|\partial^\alpha u\|_{L_p(0,T;L_q(\Omega_t))}.$$

Let us also introduce a brief notation for the regularity class of the solution. Namely, for a function g and a vector field $\mathbf{f}$ defined on $(0,T)\times\Omega_t$ we define

$$\|g,\mathbf{f}\|_{\mathcal{X}(T)} = \|\mathbf{f}\|_{L_p(0,T;W^2_q(\Omega_t))} + \|\mathbf{f}_t\|_{L_q(0,T;L_q(\Omega_t))} + \|g\|_{L_p(0,T;W^1_q(\Omega_t))} + \|g_t\|_{L_p(0,T;L_q(\Omega_t))}, \tag{1.9}$$

def:X

and for a pair $\tilde{g},\tilde{\mathbf{f}}$ defined of $(0,T)\times\Omega_0$

$$\|\tilde{g},\tilde{\mathbf{f}}\|_{\mathcal{Y}(T)} = \|\tilde{\mathbf{f}}\|_{L_p(0,T;W^2_q(\Omega_0))} + \|\tilde{\mathbf{f}}_t\|_{L_q(0,T;L_q(\Omega_0))} + \|\tilde{g}\|_{W^1_p(0,T;W^1_q(\Omega_0))}. \tag{1.10}$$

def:Y

Obviously, we denote by $\mathcal{X}(T)$ and $\mathcal{Y}(T)$ spaces for which above norms are finite.

Remark 1.1 *Notice that the norm (1.10) involves also $\|\tilde{g}_t\|_{L_p(W^1_q(\Omega_0))}$ while in (1.9) we have only $\|g_t\|_{L_p(L_q(\Omega_t))}$. The reason is that in the Lagrangian coordinates we are able to show higher regularity of the density, which does not correspond to equivalent regularity in Eulerian coordinates, see Section 4.4.*

Finally, by $E(\cdot)$ we shall denote a non-negative non-decreasing continuous function such that $E(0) = 0$.

1.2 Main results

The first main result of this paper gives the local well-posedness for system (1.1)-(1.2) with Dirichlet boundary condition.

t1 **Theorem 1.1** *Let $\Omega_0 \subset \mathbb{R}^3$ be a bounded uniform C^2 domain. Assume*

$$\varrho_0 \in W^1_q(\Omega_0), \quad \mathbf{u}_0 \in B^{2-2/p}_{q,p}(\Omega_0)$$

and

$$\mathbf{V} \in L_p(0,T;W^2_q(\mathbb{R}^3)) \cap W^1_p(0,T;L_q(\mathbb{R}^3))$$

with $2 < p < \infty$, $3 < q < \infty$ and $\frac{2}{p} + \frac{3}{q} < 1$. Then for any $L > 0$ there exists $T > 0$ such that if

$$\|\varrho_0\|_{W^1_q(\Omega_0)} + \|\mathbf{u}_0\|_{B^{2-2/p}_{q,p}(\Omega_0)} + \|\mathbf{V}\|_{L_p(0,T;W^2_q(\mathbb{R}^3))} + \|\partial_t \mathbf{V}\|_{L_p(0,T;L_q(\mathbb{R}^3))} \leq L \tag{1.11}$$ init:norr

then the system (1.1)-(1.5) admits a unique strong solution $(\varrho, \mathbf{u}) \in \mathcal{X}(T)$ and

$$\|\varrho, \mathbf{u}\|_{\mathcal{X}(T)} \leq CL. \tag{1.12}$$ est:t1

Remark 1.2 *Let us comment on the restrictions on p and q. The condition $q > 3$ is natural as we shall repeatedly use the embedding $W^1_q(\Omega_0) \subset L_\infty(\Omega_0)$. However, a stronger condition $\frac{2}{p} + \frac{3}{q} < 1$ is required since we need the embedding $B^{2(1-1/p)}_{q,p}(\Omega_0) \subset W^1_\infty(\Omega_0)$ to prove Lemma 4.2, see Corollary 3.1.*

The second main result gives global well-posedness:

t2 **Theorem 1.2** *Let $\Omega_0 \subset \mathbb{R}^3$ be a bounded uniform C^2 domain. Assume that*

$$\varrho_0 \in W^1_q(\Omega_0), \quad \mathbf{u}_0 \in B^{2-2/p}_{q,p}(\Omega_0)$$

Furthermore, let $\varrho^*, \gamma > 0$ be given constants. Then there exists $\epsilon > 0$ such that if

$$\|\varrho_0 - \varrho^*\|_{W^1_p(\Omega_0)} + \|\mathbf{u}_0 - \mathbf{V}(0)\|_{B^{2-2/p}_{q,p}(\Omega_0)} + \|e^{\gamma t}(\partial_t \mathbf{V}, \nabla_x \mathbf{V}, \nabla^2_x \mathbf{V})\|_{L_p(0,T;L_q(\mathbb{R}^3))} \leq \epsilon \tag{1.13}$$ init:norr

then the unique strong solution to (1.1)-(1.5) is defined globally in time and

$$\|e^{\gamma t}\varrho\|_{W^1_p(0,\infty;W^1_q(\Omega_t))} + \|e^{\gamma t}\partial_t \mathbf{u}\|_{L_p(0,\infty;L_q(\Omega_t))} + \|e^{\gamma t}\nabla_x \mathbf{u}\|_{L_p(0,\infty;W^1_q(\Omega_t))} + \|e^{\gamma t}(\mathbf{u} - \mathbf{V})\|_{L_p(0,\infty;L_q(\Omega_t))} \leq C\epsilon, \tag{1.14}$$ est:glob

$$\|\mathbf{u}\|_{L_p(0,\infty;L_q(\Omega_t))} \leq C\epsilon + \|\mathbf{V}\|_{L_p(0,\infty;L_q(\Omega_t))}. \tag{1.15}$$ est:glob

The paper is structured as follows. In Section 2 we rewrite the problem on a fixed domain using Lagrangian coordinates. In Section 3 we recall known results which we apply to prove Theorems 1.1 and 1.2. These are results on the existence of solutions to linearized problems on a fixed domain and certain imbedding properties. Section 4 is dedicated to the proof of Theorem 1.1. We reduce the problem to homogeneous boundary condition and show appropriate estimates of the right hand side of the problem in Lagrangian coordinates and conclude using fixed point argument and linear result recalled in Section 3. In Section 5 we prove Theorem 1.2. For this purpose we obtain appropriate estimates of the right hand side which allow to show uniform in time estimate for the solution using exponential decay property of the linear problem. This estimate allow to prolong the solution for arbitrarily large times.

2 Lagrangian transformation

Let us start with a following observation

l:lag1 **Lemma 2.1** *Let p and q satisfy the assumptions of Theorem 1.1. Then*
(i) if $\|f\|_{L_p(0,T;W^2_q(\Omega_0))} \leq M$ for some $M > 0$, then

$$\int_0^T \|\nabla f(t,\cdot)\|_{L_\infty(\Omega_0)} \mathrm{d}t \leq M E(T). \tag{2.1}$$ 2:1

(ii) if $\|e^{\gamma t} f\|_{L_p(0,\infty;W^2_q(\Omega_0))} \leq M$ for some $M, \gamma > 0$ then

$$\int_0^\infty \|\nabla f(t,\cdot)\|_{L_\infty(\Omega_0)} \mathrm{d}t \leq CM. \tag{2.2}$$ 2:1a

Proof: By the imbedding theorem and Hölder inequality we have

$$\int_0^T \|\nabla f(t,\cdot)\|_{L_\infty(\Omega_0)} \mathrm{d}t \leq C \int_0^T \|f(t,\cdot)\|_{W^2_q(\Omega_0)} \mathrm{d}t \leq T^{1/p'} \int_0^T \left(\|f(t,\cdot)\|^p_{W^2_q(\Omega_0)} \right)^{1/p} \mathrm{d}t \leq M E(T),$$

which proves the first assertion, and for the second we have

$$\begin{aligned} \int_0^\infty \|\nabla f(t,\cdot)\|_{L_\infty(\Omega_0)} \mathrm{d}t &\leq C \int_0^\infty e^{-\gamma t} e^{\gamma t} \|f(t,\cdot)\|_{W^2_q(\Omega_0)} \mathrm{d}t \\ &\leq \left(\int_0^\infty e^{-\gamma t p'} \mathrm{d}t \right)^{1/p'} \|e^{\gamma t} f\|_{L_p(0,\infty;W^2_q(\Omega_0))} \leq CM. \end{aligned}$$

□

In order to transform problem (1.1)-(1.2) to a fixed domain we introduce the change of coordinates

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{X_u}(t,y) = \mathbf{u}\Big(t, \mathbf{X_u}(t,y)\Big) \quad \text{for } t > 0, \quad \mathbf{X_u}(0,y) = y, \tag{2.3}$$ eq:coc

i.e.

$$\mathbf{X_u}(t,y) = y + \int_0^t \mathbf{u}(s, \mathbf{X_u}(s,y)) \mathrm{d}s. \tag{2.4}$$ eq:coc1

Then for any differentiable function f defined on Q_T we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t, \mathbf{X_u}(t,y)) = \frac{\partial}{\partial t} f(t, \mathbf{X_u}(t,y)) + \mathbf{u} \cdot \nabla_x f(t, \mathbf{X_u}(t,y)). \tag{2.5}$$ dt:lag

Let us define transformed density and velocities on a fixed domain Ω_0:

$$\tilde{\varrho}(t,y) = \varrho(t, \mathbf{X_u}(t,y)), \quad \tilde{\mathbf{u}}(t,y) = \mathbf{u}(t, \mathbf{X_u}(t,y)), \quad \tilde{\mathbf{V}}(t,y) = \mathbf{V}(t, \mathbf{X_u}(t,y)) \tag{2.6}$$

l:lag2 **Lemma 2.2** *Assume that*

$$\int_0^T \|\nabla_y \tilde{\mathbf{u}}\|_\infty \mathrm{d}t \leq \delta \tag{2.7}$$ small:2

for sufficiently small $\delta > 0$. Then the inverse to $\mathbf{X_u}$, i.e. $\mathbf{Y}(t,x)$ defined as

$$\mathbf{X_u}(t, \mathbf{Y}(t,x)) = x \quad \forall t \geq 0,\ x \in \Omega_t, \tag{2.8}$$ eq:inv

is well defined and its Jacobian can be expressed in a following way

$$\nabla_x \mathbf{Y}(t, \mathbf{X_u}(t,y)) = [\nabla_y \mathbf{X_u}(t,y)]^{-1} = \mathbf{I} + \mathbf{E}^0(\mathbf{k}_{\tilde{\mathbf{u}}}(t,y)), \tag{2.9}$$ 2:3

where

$$\mathbf{k}_{\tilde{\mathbf{u}}}(t,y) = \int_0^t \nabla_y \tilde{\mathbf{u}}(s,y) \mathrm{d}s \tag{2.10}$$ ku

and $\mathbf{E}^0(\cdot)$ is a 3×3 matrix of smooth functions with $\mathbf{E}^0(0) = 0$.

Proof: We have

$$\frac{\partial X_i}{\partial y_j}(t,y) = \delta_{ij} + \int_0^t \frac{\partial \tilde{u}_i}{\partial y_j}(s,y)\, ds. \tag{2.11}$$ 2:2

Therefore, if (2.7) holds for sufficiently small δ then $\mathbf{Y}(t,x)$ is well defined and we have (2.9)-(2.10) with $\mathbf{E}^0$ as in the statement of the Lemma. Next, by the boundary condition (1.4) we have

$$\mathbf{X_u}(\Gamma_0, t) = \Gamma_t \quad \text{for } t > 0$$

and

$$\mathbf{X_u}(y,t) \subset \Omega_t \quad \text{for } t > 0,\ y \in \Omega_0.$$

Finally, it is well known that $\mathbf{X_u}$ is a diffeomorphism which completes the proof.

□

Note that by (2.9) we can write

$$\nabla_x = [\mathbf{I} + \mathbf{E}^0(\mathbf{k}_{\tilde{\mathbf{u}}})]\nabla_y. \tag{2.12}$$ dx:lag

Lemma 2.3 *Let $(\varrho, \mathbf{u})$ be a solution to (1.1)-(1.5). Then $(\tilde{\varrho}, \tilde{\mathbf{u}})$ solve the following system of equations on the fixed domain Ω_0*

$$\tilde{\varrho}\tilde{\mathbf{u}}_t - \mu\Delta_y \tilde{\mathbf{u}} - \left(\frac{\mu}{3} + \zeta\right)\nabla_y \mathrm{div}\,_y \tilde{\mathbf{u}} + \nabla_y p(\tilde{\varrho}) = \mathbf{F}(\tilde{\varrho}, \tilde{\mathbf{u}}) \tag{2.13}$$ ME:lag1

$$\tilde{\varrho}_t + \tilde{\varrho}\mathrm{div}\,_y \tilde{\mathbf{u}} = G(\tilde{\varrho}, \tilde{\mathbf{u}}), \tag{2.14}$$ CE:lag1

$$\tilde{\mathbf{u}}|_{t=0} = \mathbf{u}_0, \quad \tilde{\mathbf{u}}|_{\partial\Omega_0} = \tilde{\mathbf{V}}. \tag{2.15}$$ icbc:lag

The i-th component of $\mathbf{F}(\cdot,\cdot)$ is given by

$$F_i(\tilde{\varrho}, \tilde{\mathbf{u}}) = -E^0_{ij}\partial_{y_j} p(\tilde{\varrho}) + R_i(\tilde{\mathbf{u}}), \tag{2.16}$$ F:lag1

and

$$G(\tilde{\varrho}, \tilde{\mathbf{u}}) = -\tilde{\varrho}\mathbf{E}^0_{ij}(\mathbf{k}_{\tilde{\mathbf{u}}})\frac{\partial \tilde{u}_i}{\partial y_j}, \tag{2.17}$$ G:lag1

where the components $R_i(\cdot)$ of $\mathbf{R}(\cdot)$ are expressed as

$$R_i(\tilde{\mathbf{u}}) = \mu[A_{2\Delta}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y^2 \tilde{\mathbf{u}} + A_{1\Delta}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y \tilde{\mathbf{u}}]_i + \left(\frac{\mu}{3} + \zeta\right)[A_{2\mathrm{div}\,,i}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y^2 \tilde{\mathbf{u}} + A_{1\mathrm{div}\,,i}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y \tilde{\mathbf{u}}], \tag{2.18}$$ def:R

with $A_{j\Delta}$ and $A_{j\mathrm{div}}$ ($j = 1,2$) given in (2.23), (2.24), (2.26) and (2.27), respectively.

Proof: We have

$$\mathrm{div}_{\,x}\mathbf{u} = \mathrm{div}_{\,y}\tilde{\mathbf{u}} + \mathbf{E}^0 : \nabla_y \tilde{\mathbf{u}}, \tag{2.19}$$ lag:div

where $\mathbf{E}^0 : \nabla_y \tilde{\mathbf{u}} = \mathbf{E}^0_{ij}(\mathbf{k}_{\tilde{\mathbf{u}}})\frac{\partial \tilde{u}_i}{\partial y_j}$, which together with (2.5) gives (2.14).

In order to transform the momentum equation (1.2) it is convenient to rewrite it, using (1.1) and (1.3), as

$$\varrho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla_x \mathbf{u}) - \mu\Delta_x \mathbf{u} - (\frac{\mu}{3} + \zeta)\nabla \mathrm{div}_{\,x}\mathbf{u} + \nabla_x p(\varrho) = 0. \tag{2.20}$$ i1bb

We have

$$\partial_{x_i} p(\varrho) = \partial_{y_i} p(\tilde{\varrho}) + E^0_{ij}\partial_{y_j} p(\tilde{\varrho}). \tag{2.21}$$ lag:5

Now we need to transform second order operators. By (2.12), we have

$$\Delta_x \mathbf{u} = \frac{\partial}{\partial x_k}\left(\frac{\partial \mathbf{u}}{\partial x_k}\right) = \left(\delta_{kl} + \mathbf{E}^0_{kl}(\mathbf{k}_{\tilde{\mathbf{u}}})\right)\frac{\partial}{\partial y_l}\left(\left(\delta_{km} + \mathbf{E}^0_{km}(\mathbf{k}_{\tilde{\mathbf{u}}})\right)\frac{\partial \tilde{\mathbf{u}}}{\partial y_m}\right).$$

Therefore

$$\Delta_x \mathbf{u} = \Delta_y \tilde{\mathbf{u}} + A_{2\Delta}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y^2 \tilde{\mathbf{u}} + A_{1\Delta}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y \tilde{\mathbf{u}} \tag{2.22}$$ lag:6

with

$$A_{2\Delta}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y^2 \tilde{\mathbf{u}} = 2\sum_{l,m}\mathbf{E}^0_{kl}(\mathbf{k}_{\tilde{\mathbf{u}}})\frac{\partial^2 \tilde{\mathbf{u}}}{\partial y_l \partial y_m} + \sum_{k,l,m}\mathbf{E}^0_{kl}(\mathbf{k}_{\tilde{\mathbf{u}}})\mathbf{E}^0_{km}(\mathbf{k}_{\tilde{\mathbf{u}}})\frac{\partial^2 \tilde{\mathbf{u}}}{\partial y_l \partial y_m}, \tag{2.23}$$ a2delta

$$\begin{aligned} A_{1\Delta}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y \tilde{\mathbf{u}} =& (\nabla_{\mathbf{k}_{\tilde{\mathbf{u}}}}\mathbf{E}^0_{lm})(\mathbf{k}_{\tilde{\mathbf{u}}})\int_0^t (\partial_l \nabla_y \tilde{\mathbf{u}})\,\mathrm{d}s\frac{\partial \tilde{\mathbf{u}}}{\partial y_m} \\ &+ \mathbf{E}^0_{kl}(\mathbf{k}_{\tilde{\mathbf{u}}})(\nabla_{\mathbf{k}_{\tilde{\mathbf{u}}}}\mathbf{E}^0_{km})(\mathbf{k}_{\tilde{\mathbf{u}}})\int_0^t \partial_l \nabla_y \tilde{\mathbf{u}}\,\mathrm{d}s\frac{\partial \tilde{\mathbf{u}}}{\partial y_m}. \end{aligned} \tag{2.24}$$ a1delta

Next, by (2.19)

$$\frac{\partial}{\partial x_i}\mathrm{div}_{\,x}\mathbf{u} = \sum_{k=1}^{3}(\delta_{ik} + \mathbf{E}^0_{ik}(\mathbf{k}_{\tilde{\mathbf{u}}}))\frac{\partial}{\partial y_k}\left(\mathrm{div}_y \tilde{\mathbf{u}} + \sum_{l,m=1}^{3}\mathbf{E}^0_{lm}(\mathbf{k}_{\tilde{\mathbf{u}}})\frac{\partial \tilde{u}_l}{\partial y_m}\right),$$

so we obtain

$$\frac{\partial}{\partial x_i}\mathrm{div}_x \mathbf{u} = \frac{\partial}{\partial y_i}\mathrm{div}_y \tilde{\mathbf{u}} + A_{2\mathrm{div}\,,i}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y^2 \tilde{\mathbf{u}} + A_{1\mathrm{div}\,,i}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y \tilde{\mathbf{u}}, \tag{2.25}$$ lag:7

where

$$A_{2\mathrm{div}\,,i}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y^2 \tilde{\mathbf{u}} = \sum_{l,m=1}^{3}\mathbf{E}^0_{lm}(\mathbf{k}_{\tilde{\mathbf{u}}})\frac{\partial^2 \tilde{u}_l}{\partial y_m \partial y_i} + \sum_{k=1}^{3}\mathbf{E}^0_{ik}(\mathbf{k}_{\tilde{\mathbf{u}}})\frac{\partial}{\partial y_k}\mathrm{div}_y \tilde{\mathbf{u}} + \sum_{k,l=1}^{3}\mathbf{E}^0_{ik}(\mathbf{k}_{\tilde{\mathbf{u}}})\mathbf{E}^0_{lm}(\mathbf{k}_{\tilde{\mathbf{u}}})\frac{\partial^2 \tilde{u}_l}{\partial y_k \partial y_m}, \tag{2.26}$$ a2div

$$\begin{aligned} A_{1\mathrm{div}\,,i}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y \tilde{\mathbf{u}} =& \sum_{l,m=1}^{3}(\nabla_{\mathbf{k}_{\tilde{\mathbf{u}}}}\mathbf{E}^0_{lm})(\mathbf{k}_{\tilde{\mathbf{u}}})\int_0^t \partial_i \nabla_y \tilde{\mathbf{u}}\,\mathrm{d}s\frac{\partial \tilde{u}_l}{\partial y_m} \\ &+ \sum_{k,l,m=1}^{3}\mathbf{E}^0_{ik}(\mathbf{k}_{\tilde{\mathbf{u}}})(\nabla_{\mathbf{k}_{\tilde{\mathbf{u}}}}\mathbf{E}^0_{lm})(\mathbf{k}_{\tilde{\mathbf{u}}})\int_0^t \partial_k \nabla_y \tilde{\mathbf{u}}\,\mathrm{d}s\frac{\partial \tilde{u}_l}{\partial y_m}. \end{aligned} \tag{2.27}$$ a1div

Putting together (2.21), (2.22) and (2.25) gives (2.13) with (2.16).

3 Linear theory and auxiliary results

First we recall a maximal regularity result concerning the linear problem on a fixed domain, which will be used in the proof of Theorem 1.1. The linearized system of equations on the fixed domain Ω_0 reads as

$$\varrho_0 \mathbf{u}_t - \mu \Delta_y \mathbf{u} - (\frac{\mu}{3} + \zeta) \nabla_y \mathrm{div}_y \mathbf{u} + \gamma \nabla_y \eta = \mathbf{f}, \tag{3.1}$$
ME:lin0

$$\eta_t + \varrho_0 \mathrm{div}_y \mathbf{u} = g, \tag{3.2}$$
CE:lin0

$$\mathbf{u}|_{\partial\Omega_0} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0. \tag{3.3}$$
icbc:lin0

To show the local well-posedness for the Dirichlet boundary condition we will use the following result

p1 **Proposition 3.1** *Let $1 < p, q < \infty$ and $\frac{2}{p} + \frac{1}{q} \neq 1$. Let $\varrho_0, \mathbf{u}_0, \mu$ and ζ satisfy the assumptions of Theorem 1.1. Moreover, let $\Omega_0 \subset \mathbb{R}^n$ be a uniform C^2 domain. If $\frac{2}{p} + \frac{1}{q} < 1$, assume additionally that the initial velocity satisfies the compatibility condition $\mathbf{u}_0|_{\partial\Omega_0} = 0$. Finally, assume that for some $T > 0$*

$$\mathbf{f} \in L_p(0,T; L_q(\Omega_0)), \quad g \in L_p(0,T; W^1_q(\Omega_0)).$$

Then the problem (3.1)-(3.3) admits a unique solution $(\varrho, \mathbf{u}) \in \mathcal{Y}(T)$ such that

$$\|\varrho, \mathbf{u}\|_{\mathcal{Y}(T)} \leq C(T, \mu, \zeta, \|\varrho_0\|_{L_\infty(\Omega_0)})[\|\mathbf{u}_0\|_{B^{2-2/p}_{q,p}(\Omega_0)} + \|\mathbf{f}\|_{L_p(0,T;L_q(\Omega_0)} + \|g\|_{L_p(0,T;W^1_q(\Omega_0)}]. \tag{3.4}$$
est:linMI

In order to show the global well-posedness in Theorem 1.2 we will linearize the problem around the constant ϱ^*, therefore we consider on the fixed domain Ω_0 a linear problem

$$\varrho^* \mathbf{u}_t - \mu \Delta_y \mathbf{u} - (\frac{\mu}{3} + \zeta) \nabla_y \mathrm{div}_y \mathbf{u} + \gamma \nabla_y \eta = \mathbf{f}, \tag{3.5}$$
ME:lin1a

$$\eta_t + \varrho^* \mathrm{div}_y \mathbf{u} = g, \tag{3.6}$$
CE:lin1a

$$\mathbf{u}|_{\partial\Omega_0} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0. \tag{3.7}$$
icbc:lin

We have the following exponential decay estimate

p2 **Proposition 3.2** *Let $1 < p, q < \infty$. Let $\Omega_0 \in \mathbb{R}^n$ be bounded, uniform C^2 domain. Assume $p, q, \mu, \zeta, \mathbf{u}_0$ satisfy the assumptions of Proposition 3.1. Assume moreover that there exist $\gamma > 0$ such that*

$$e^{\gamma t}\mathbf{f} \in L_p(0,\infty; L_q(\Omega_0)), \quad e^{\gamma t} g \in W^1_p(0,\infty; L_q(\Omega_0)).$$

Then (3.5)-(3.7) admits a unique solution $\varrho, \mathbf{u}$ such that

$$\begin{aligned} &\|e^{\gamma t}\mathbf{u}_t\|_{L_p(0,\infty;L_q(\Omega_0))} + \|e^{\gamma t}\mathbf{u}\|_{L_p(0,\infty;W^2_q(\Omega_0))} + |\gamma|^{1/2}\|e^{\gamma t}\nabla_y \mathbf{u}\|_{L_p(0,\infty;L_q(\Omega_0))} + \|e^{\gamma t}\eta\|_{W^1_p(0,\infty;W^1_q(\Omega_0))} \\ &\leq C_{p,q}\left(\|\mathbf{u}_0\|_{B^{2-2/p}_{q,p}(\Omega_0)} + \|e^{\gamma t}\mathbf{f}\|_{L_p(0,\infty;L_q(\Omega_0))} + \|e^{\gamma t} g\|_{W^1_q(0,\infty;L_q(\Omega_0))}\right). \end{aligned} \tag{3.8}$$

Remark 3.1 *Propositions 3.1 and 3.2 can be deduced directly from, respectively, Theorems 4.1 and 5.1 in [23] as their special cases. Alternatively, they can be deduced from Theorems 2.8 and 2.9 in [3], respectively.*

Next we recall some embedding results for Besov spaces. The first one is [1, Theorem 7.34 (c)]:

:imbed1 **Lemma 3.1** *Assume $\Omega \in \mathbb{R}^n$ satisfies the cone condition and let $1 \le p, q \le \infty$ and $sq > n$. Then*

$$B^s_{q,p}(\Omega) \subset C_B(\Omega),$$

where C_B we denote the space of continuous bounded functions.

In particular $u \in B^{2-2/p}_{q,p}(\Omega_0)$ implies $\nabla u \in B^{1-2/p}_{q,p}(\Omega_0)$. Therefore the above Lemma with $s = 1 - 2/p$ yields

:imbed1 **Corollary 3.1** *Assume $\frac{2}{p} + \frac{3}{q} < 1$ and let Ω_0 satisfy the assumptions of Theorem 1.1. Then $B^{2-2/p}_{q,p}(\Omega_0) \subset W^1_\infty(\Omega_0)$ and*

$$\|f\|_{W^1_\infty(\Omega_0)} \le C\|f\|_{B^{2-2/p}_{q,p}(\Omega_0)}. \tag{3.9}$$

The next result is due to Tanabe (cf. [27, p.10]):

L:int **Lemma 3.2** *Let X and Y be two Banach spaces such that X is a dense subset of Y and $X \subset Y$ is continuous. Then for each $p \in (1, \infty)$*

$$W^1_p((0,\infty), Y) \cap L_p((0,\infty), X) \subset C([0,\infty), (X,Y)_{1/p,p})$$

and for every $u \in H^1_p((0,\infty), Y) \cap L_p((0,\infty), X)$ we have

$$\sup_{t\in(0,\infty)} \|u(t)\|_{(X,Y)_{1/p,p}} \le (\|u\|^p_{L_p((0,\infty),X)} + \|u\|^p_{W^1_p((0,\infty),Y)})^{1/p}.$$

□

4 Local well-posedness

4.1 Linearization for the local well-posedness

Let us start with removing inhomogeneity from the boundary condition (2.15). For this purpose we show

Lemma 4.1 *Let $\mathbf{V}$ satisfy the assumptions of Theorem 1.1. Then the problem*

$$\begin{aligned} &\varrho_0 \partial_t \mathbf{u}_{b1} - \mu\Delta_y \mathbf{u}_{b1} - (\frac{\mu}{3} + \zeta)\nabla_y \mathrm{div}\,_y \mathbf{u}_{b1} = 0 \qquad \text{in } \Omega_0 \times (0,T), \\ &\mathbf{u}_{b1}|_{\Gamma_0} = \tilde{\mathbf{V}}, \quad \mathbf{u}_{b1}|_{t=0} = \mathbf{V}(0) \end{aligned} \tag{4.1}$$ def:vub

admits a unique solution such that

$$\|\partial_t \mathbf{u}_{b1}\|_{L_p(0,T;L_q(\Omega_0))} + \|\mathbf{u}_{b1}\|_{L_p(0,T;W^2_q(\Omega_0))} \le C\|\partial_t \tilde{\mathbf{V}}\|_{L_p(0,T;L_q(\Omega_0))} + \|\tilde{\mathbf{V}}\|_{L_p(0,T;W^2_q(\Omega_0))}. \tag{4.2}$$ ub:loc

Proof. Denoting $\tilde{\mathbf{u}}_{b1} = \mathbf{u}_{b1} - \tilde{\mathbf{V}}$ we have

$$\begin{aligned} &\varrho_0 \partial_t \tilde{\mathbf{u}}_{b1} - \mu\Delta_y \tilde{\mathbf{u}}_{b1} - (\frac{\mu}{3} + \zeta)\nabla_y \mathrm{div}\,_y \tilde{\mathbf{u}}_{b1} = \varrho_0 \partial_t \tilde{\mathbf{V}} - \mu\Delta_y \tilde{\mathbf{V}} - (\frac{\mu}{3} + \zeta)\nabla_y \mathrm{div}\,_y \tilde{\mathbf{V}} \qquad \text{in } \Omega_0 \times (0,T), \\ &\tilde{\mathbf{u}}_{b1}|_{\Gamma_0} = 0, \quad \tilde{\mathbf{u}}_{b1}|_{t=0} = 0. \end{aligned} \tag{4.3}$$ def:vuba

Therefore, if $\mathbf{V}$ satisfies the assumptions of Theorem 1.1 then a maximal regularity result for the momentum equation, which can be deduced similarly to Proposition 3.1, gives

$$\|\partial_t \tilde{\mathbf{u}}_{b1}\|_{L_p(0,T;L_q(\Omega_0))} + \|\tilde{\mathbf{u}}_{b1}\|_{L_p(0,T;W^2_q(\Omega_0))} \le C\|\partial_t \tilde{\mathbf{V}}, \nabla^2_y \tilde{\mathbf{V}}\|_{L_p(0,T;L_q(\Omega_0))}, \tag{4.4}$$

which implies (4.2).

□

As linear system in Proposition 3.1 has constant in time coefficients, we linearize (2.13)-(2.14) around the initial condition. Denoting

$$\eta = \tilde{\varrho} - \varrho_0, \quad \mathbf{v} = \tilde{\mathbf{u}} - \mathbf{u}_{b1}$$

we obtain

$$\varrho_0 \mathbf{v}_t - \mu \Delta_y \mathbf{v} - \left(\frac{\mu}{3} + \zeta\right) \nabla_y \mathrm{div}_y \mathbf{v} + \gamma_1 \nabla_y \eta = \mathbf{F}_1(\eta, \mathbf{v}) \tag{4.5}$$ ME:lin1

$$\eta_t + \varrho_0 \mathrm{div}_y \mathbf{v} = G_1(\eta, \mathbf{v}), \tag{4.6}$$ CE:lin1

$$\mathbf{v}|_{t=0} = \mathbf{u}_0 - \mathbf{V}(0), \quad \mathbf{v}|_{\partial\Omega_0} = 0. \tag{4.7}$$ icbc:lin

where $\gamma_1 = p'(\varrho_0)$ and

$$\mathbf{F}_1(\eta, \mathbf{v}) = \mathbf{R}(\eta + \varrho_0, \mathbf{v} + \mathbf{u}_{b1}) - \eta \partial_t (\mathbf{v} + \mathbf{u}_{b1}) - p'(\varrho_0) \nabla_y \varrho_0 - [p'(\eta + \varrho_0) - p'(\varrho_0)] \nabla_y \eta \tag{4.8}$$ F:lin1

$$G_1(\eta, \mathbf{v}) = G(\eta + \varrho_0, \mathbf{v} + \mathbf{u}_{b1}) - (\eta + \varrho_0) \mathrm{div}_y \mathbf{u}_{b1} - \eta \mathrm{div}_y \mathbf{v}, \tag{4.9}$$ G:lin1

and $\mathbf{R}(\tilde{\varrho}, \tilde{\mathbf{u}})$ is defined in (2.18).

4.2 Nonlinear estimates for the local well-posedness.

Using the results recalled in the previous section we show the following estimate for functions from the space $\mathcal{Y}(T)$:

nonlin2 **Lemma 4.2** *Let* $(\tilde{\varrho}, \tilde{\mathbf{u}}) \in B(0, M) \subset \mathcal{Y}(T)$ *and let* $\mathbf{u}_0$ *satisfy the assumptions of Theorem 1.1. Then*

$$\|\mathbf{E}^0(\mathbf{k}_{\tilde{\mathbf{u}}}), \nabla_{\mathbf{k}_{\tilde{\mathbf{u}}}} \mathbf{E}^0(\mathbf{k}_{\tilde{\mathbf{u}}}), \|_{L_\infty((0,T)\times\Omega_0)} \leq C(M, L) E(T), \tag{4.10}$$ est:01

$$\sup_{t\in(0,T)} \|\eta(\cdot, t)\|_{W^1_q(\Omega_0)} \leq C(M, L) E(T), \tag{4.11}$$ est:02

$$\sup_{t\in(0,T)} \|\mathbf{v}(\cdot, t) - \mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}} \leq C(M, L), \tag{4.12}$$ est:03

$$\|\mathbf{v}\|_{L_\infty(0,T;W^1_\infty(\Omega_0))} \leq C(M, L), \tag{4.13}$$ est:04

where $\mathbf{k}_{\tilde{\mathbf{u}}}$ *is defined in* (2.10).

Proof. This result is in fact part of [24, Lemma 5.6]. For the sake of completeness we present an outline of the proof referring there for details. First, (4.10) follows immediately from Lemma 2.1. In order to prove (4.12) we extend $\mathbf{v} - \mathbf{u}_0$ to the whole real line in time and apply Lemma 3.2 with $X = W^1_q(\Omega_0)$ and $Y = L_q(\Omega_0)$. Finally (4.13) follows from (4.12) and Corollary 3.1.

□

From the proof of (4.13) we can also deduce

nonlin1 **Lemma 4.3** *Let* $f_t \in L_p(0, T; L_q(\Omega_0))$, $f \in L_p(0, T; W^2_q(\Omega_0))$ $f(0, \cdot) \in B^{2-2/p}_{q,p}(\Omega_0)$. *Then*

$$\|f\|_{L_\infty(0,T,W^1_\infty(\Omega_0))} \leq C[\|f_t\|_{L_p(0,T;L_q(\Omega_0))} + \|f\|_{L_p(0,T;W^2_q(\Omega_0))} + \|f(0)\|_{B^{2-2/p}_{q,p}(\Omega_0)}]. \tag{4.14}$$

Now we can estimate the right hand side of (2.13) in the regularity required by Proposition 3.1:

nonlin3 **Lemma 4.4** *Let* $\mathbf{F}_1(\eta, \mathbf{v}), G_1(\eta, \mathbf{v})$ *be defined in* (4.8) *and* (4.9). *Assume that* $\varrho_0, \mathbf{u}_0$ *and* $\mathbf{V}$ *satisfy* (1.11). *Then*

$$\|\mathbf{F}_1(\eta, \mathbf{v})\|_{L_p(0,T;L_q(\Omega_0))} + \|G_1(\eta, \mathbf{v})\|_{L_p(0,T;W^1_q(\Omega_0))} \leq E(T)(\|\eta, \mathbf{v}\|_{\mathcal{Y}(T)} + L). \tag{4.15}$$ est:FG:l

Proof: The proof relies on the estimates collected in Lemma 4.2. By (4.11) and (4.2) we have

$$\begin{aligned}\|\eta\partial_t(\mathbf{v}+\mathbf{u}_{b1})\|_{L_p(0,T;L_q(\Omega_0))} &\leq \|\eta\|_{L_\infty(\Omega_0\times(0,T))}\left(\|\partial_t\mathbf{v}\|_{L_p(0,T;L_q(\Omega_0))}+\|\partial_t\mathbf{u}_{b1}\|_{L_p(0,T;L_q(\Omega_0))}\right) \\ &\leq E(T)[\|\eta,\mathbf{v}\|_{\mathcal{Y}(T)}+L].\end{aligned} \tag{4.16}$$

In order to estimate the remaining terms notice that all the quantities (2.23)-(2.27) contain either $\mathbf{E}(\mathbf{k}_{\tilde{\mathbf{u}}})$ or $\nabla\mathbf{E}(\mathbf{k}_{\tilde{\mathbf{u}}})$ multiplied by the derivatives of $\tilde{\mathbf{u}}$ with respect to y of at most second order. Therefore (4.10) and (4.2) imply

$$\|A_{2\Delta}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y^2\tilde{\mathbf{u}},\ A_{1\Delta}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y\tilde{\mathbf{u}},\ A_{2\mathrm{div}\,,i}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y^2\tilde{\mathbf{u}},\ A_{1\mathrm{div}\,,i}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y\tilde{\mathbf{u}}\|_{L_p(0,T;L_q(\Omega_0))}\leq E(T)[\|\eta,\mathbf{v}\|_{\mathcal{Y}(T)}+L]. \tag{4.17}$$
est:A

Putting together all above estimates we get the estimate for $\mathbf{F}_1$. Next, (4.10) gives immediately

$$\|G(\tilde{\varrho},\tilde{\mathbf{u}})\|_{L_p(0,T;W_q^1(\Omega_0))}\leq E(T)[\|\eta,\mathbf{v}\|_{\mathcal{Y}(T)}+L],$$

and thus (4.15) follows.

□

4.3 Fixed point argument

Let us define a solution operator

$$(\eta,\mathbf{v})=S(\bar{\eta},\bar{\mathbf{v}})\iff(\eta,\mathbf{v})\text{ solves (4.5)-(4.7) with right hand side }\mathbf{F}_1(\bar{\eta},\bar{\mathbf{v}}),G_1(\bar{\eta},\bar{\mathbf{v}}).$$

By Proposition 3.1 and Lemma 4.4, S is well defined on $\mathcal{Y}(T)$ and maps a ball $B(0,M)\subset\mathcal{Y}(T)$ into itself provided T is sufficiently small w.r.t. M and L. Denote

$$(\eta_i,\mathbf{v}_i)=S(\bar{\eta}_i,\bar{\mathbf{v}}_i),\quad i=1,2.$$

Then the difference $(\eta_1-\eta_2,\mathbf{v}_1-\mathbf{v}_2)$ satisfies

$$\varrho_0\partial_t(\mathbf{v}_1-\mathbf{v}_2)-\mu\Delta_y(\mathbf{v}_1-\mathbf{v}_2)-\left(\frac{\mu}{3}+\zeta\right)\nabla_y\mathrm{div}\,_y(\mathbf{v}_1-\mathbf{v}_2)+\gamma_1\nabla_y(\eta_1-\eta_2)=\mathbf{F}_1(\bar{\eta}_1,\bar{\mathbf{v}}_1)-\mathbf{F}_1(\bar{\eta}_2,\bar{\mathbf{v}}_2) \tag{4.18}$$
ME:dif

$$\partial_t(\eta_1-\eta_2)+\varrho_0\mathrm{div}\,_y(\mathbf{v}_1-\mathbf{v}_2)=G_1(\bar{\eta}_1,\bar{\mathbf{v}}_1)-G_1(\bar{\eta}_2,\bar{\mathbf{v}}_2), \tag{4.19}$$
CE:dif

$$(\mathbf{v}_1-\mathbf{v}_2)|_{t=0}=0,\quad(\mathbf{v}_1-\mathbf{v}_2)|_{\partial\Omega_0}=0, \tag{4.20}$$
icbc:dif

and we have

$$\begin{aligned}\mathbf{F}_1(\bar{\eta}_1,\bar{\mathbf{v}}_1)-\mathbf{F}_1(\bar{\eta}_2,\bar{\mathbf{v}}_2)&=\mathbf{R}(\bar{\eta}_1+\varrho_0,\bar{\mathbf{v}}_1+\mathbf{u}_{b1})-\mathbf{R}(\bar{\eta}_2+\varrho_0,\bar{\mathbf{v}}_2+\mathbf{u}_{b1})-(\bar{\eta}_1-\bar{\eta}_2)\partial_t\mathbf{u}_{b1}-\bar{\eta}_1\partial_t(\bar{\mathbf{v}}_1-\bar{\mathbf{v}}_2)\\&-\partial_t\bar{\mathbf{v}}_2(\bar{\eta}_1-\bar{\eta}_2)+p'(\varrho_0)\nabla_y(\bar{\eta}_1-\bar{\eta}_2)-p'(\bar{\eta}_1+\varrho_0)\nabla_y(\bar{\eta}_1-\bar{\eta}_2)-\nabla_y\bar{\eta}_2[p'(\bar{\eta}_1+\varrho_0)-p'(\bar{\eta}_2+\varrho_0)]\end{aligned} \tag{4.21}$$
dif:F

and

$$\begin{aligned}G_1(\bar{\eta}_1,\bar{\mathbf{v}}_1)-G_1(\bar{\eta}_2,\bar{\mathbf{v}}_2)&=-(\bar{\eta}_1-\bar{\eta}_2)\mathrm{div}\,_y\mathbf{u}_{b1}-\bar{\eta}_2\mathrm{div}\,_y(\bar{\mathbf{v}}_1-\bar{\mathbf{v}}_2)-(\bar{\eta}_1-\bar{\eta}_2)\mathrm{div}\,_y\bar{\mathbf{v}}_1-(\bar{\eta}_1+\varrho_0)\mathbf{E}^1:\nabla_y(\bar{\mathbf{v}}_1-\bar{\mathbf{v}}_2)\\&-\nabla_y(\bar{\mathbf{v}}_2+\mathbf{u}_{b1}):[(\bar{\eta}_1+\varrho_0)(\mathbf{E}^1-\mathbf{E}^2)+(\bar{\eta}_1-\bar{\eta}_2)\mathbf{E}^2],\end{aligned} \tag{4.22}$$

where we have denoted

$$\mathbf{E}^1=\mathbf{E}^0(\mathbf{k}_{\bar{\mathbf{v}}_1+\mathbf{u}_{b1}}),\quad\mathbf{E}^2=\mathbf{E}^0(\mathbf{k}_{\bar{\mathbf{v}}_2+\mathbf{u}_{b1}}).$$

Since $\mathbf{E}^0(\cdot)$ is smooth, we have

$$|\mathbf{E}^1 - \mathbf{E}^2| \le C|\mathbf{k}_{\bar{\mathbf{v}}_1+\mathbf{u}_{b1}} - \mathbf{k}_{\bar{\mathbf{v}}_2+\mathbf{u}_{b1}}| \le C\mathbf{k}_{\bar{\mathbf{v}}_1-\bar{\mathbf{v}}_2}.$$

Therefore, recalling the definition of $\mathbf{R}$ we obtain

$$\|\mathbf{R}(\bar{\eta}_1 + \varrho_0, \bar{\mathbf{v}}_1 + \mathbf{u}_{b1}) - \mathbf{R}(\bar{\eta}_2 + \varrho_0, \bar{\mathbf{v}}_2 + \mathbf{u}_{b1})\|_{L_p(0,T;L_q(\Omega_0))} \le E(T)\|(\bar{\eta}_1 - \bar{\eta}_2, \bar{\mathbf{v}}_1 - \bar{\mathbf{v}}_2)\|_{\mathcal{Y}(T)}.$$

Estimating the remaining terms on the right hand side of (4.21) similarly as in the proof of Lemma 4.4 we obtain

$$\|\mathbf{F}_1(\bar{\eta}_1, \bar{\mathbf{v}}_1) - \mathbf{F}_1(\bar{\eta}_2, \bar{\mathbf{v}}_2)\|_{L_p(0,T;L_q(\Omega_0))} \le E(T)\|(\bar{\eta}_1 - \bar{\eta}_2, \bar{\mathbf{v}}_1 - \bar{\mathbf{v}}_2)\|_{\mathcal{Y}(T)} \tag{4.23}$$

est:dif:

In a similar way we get

$$\|G_1(\bar{\eta}_1, \bar{\mathbf{v}}_1) - G_2(\bar{\eta}_2, \bar{\mathbf{v}}_2)\|_{W^1_p(0,T;L_q(\Omega_0))} \le E(T)\|(\bar{\eta}_1 - \bar{\eta}_2, \bar{\mathbf{v}}_1 - \bar{\mathbf{v}}_2)\|_{\mathcal{Y}(T)}. \tag{4.24}$$

est:dif:

Applying (4.23), (4.24) and Proposition 3.1 to system (4.18)-(4.20) we see that S is a contraction on $B(0,M) \subset \mathcal{Y}(T)$ for sufficiently small times. Therefore it has a unique fixed point $(\eta^*, \mathbf{v}^*)$. Now

$$\tilde{\varrho} = \eta^* + \varrho_0,\ \tilde{\mathbf{u}} = \mathbf{v}^* + \mathbf{u}_{b1}$$

is a solution to (2.13)-(2.15) and

$$\|\tilde{\varrho}, \tilde{\mathbf{u}}\|_{\mathcal{Y}(\mathcal{T})} \le CL.$$

It is quite standard to verify that after coming back to Eulerian coordinates we obtain a solution with the estimate (1.12), however for the sake of completeness we justify it briefly in the next subsection.

4.4 Equivalence of norms in Lagrangian and Eulerian coordinates

ec:equiv

By (4.13), the Jacobian of the transformation $\mathbf{X}_\mathbf{u}$ is bounded in space-time. Therefore, Lemma 2.2 implies the equivalence of $L_p(0,T;L_q)$ norms of a function and its first-order space derivatives. Furthermore, we have

$$\nabla^2_y \tilde{f}(t,y) = \nabla_x f(t, \mathbf{X}_\mathbf{u}(t,y))\nabla^2_y \mathbf{X}_\mathbf{u} + (\nabla_y \mathbf{X}_\mathbf{u})^2 \nabla^2_x f(t, \mathbf{X}_\mathbf{u}(t,y)).$$

Again by (4.13), $\nabla_y \mathbf{X}_\mathbf{u}$ is bounded in space-time, which together with embedding $W^1_q(\Omega_t) \subset L_\infty(\Omega_t)$ for $t \in [0,T)$ gives equivalence of $L_p(L_q)$ norms of second space derivatives. However, we have a different situation for the time derivative. The solution constructed in Lagrangian coordinates satisfies

$$\tilde{\varrho}_t \in L_p(0,T;W^1_q(\Omega_0)).$$

However, due to (2.5) this does not imply the same regularity for the density in Eulerian coordinates. Nevertheless, the regularity of $\mathbf{u}$ implies

$$\varrho_t \in L_p(0,T;L_q(\Omega_t)),$$

which is the regularity in the assertion of Theorem 1.1.

5 Global well-posedness

5.1 Linearization

Again we first reduce the problem to homogeneous boundary condition.

Lemma 5.1 *If* $\mathbf{V}$ *satisfies the assumptions of Theorem 1.2 then the problem*

$$\varrho^* \partial_t \mathbf{u}_{b2} - \mu \Delta_y \mathbf{u}_{b2} - (\frac{\mu}{3} + \zeta) \nabla_y \mathrm{div}\,_y \mathbf{u}_{b2} = 0 \qquad \text{in } \Omega_0 \times (0,T), \tag{5.1}$$

def:vub2

$$\mathbf{u}_{b2}|_{\Gamma_0} = \tilde{\mathbf{V}}, \quad \mathbf{u}_{b2}|_{t=0} = \mathbf{V}(0)$$

admits a unique global in time solutions $\mathbf{u}_{b2}$ *with the decay estimate*

$$\|e^{\gamma t}\partial_t \mathbf{u}_{b2}\|_{L_p(0,T;L_q(\Omega_0))} + \|e^{\gamma t}\nabla_y \mathbf{u}_{b2}\|_{L_p(0,T;W^1_q(\Omega_0))} + \|e^{\gamma t}(\mathbf{u}_{b2} - \tilde{\mathbf{V}}))\|_{L_p(0,T;L_q(\Omega_0))} \leq C\|e^{\gamma t}(\partial_t \tilde{\mathbf{V}}, \nabla^2_y \tilde{\mathbf{V}})\|_{L_p(0,T;L_q(\Omega_0))}. \tag{5.2}$$

ub:glob

Proof. Let us define $\tilde{\mathbf{u}}_{b_2} = \mathbf{u}_{b2} - \tilde{\mathbf{V}}$. If $\mathbf{V}$ satisfies the assumptions of Theorem 1.2, we have a decay estimate analogous to Proposition 3.2:

$$\|e^{\gamma t}\partial_t \tilde{\mathbf{u}}_{b2}\|_{L_p(0,T;L_q(\Omega_0))} + \|e^{\gamma t}\tilde{\mathbf{u}}_{b2}\|_{L_p(0,T;W^2_q(\Omega_0))} \leq C\|e^{\gamma t}(\partial_t \tilde{\mathbf{V}}, \nabla^2_y \tilde{\mathbf{V}})\|_{L_p(0,T;L_q(\Omega_0))}, \tag{5.3}$$

which gives (5.2).

□

This time we have to linearize the density around the constant ϱ^*. Denoting

$$\sigma = \tilde{\varrho} - \varrho^*, \quad \mathbf{v} = \tilde{\mathbf{u}} - \mathbf{u}_{b2}$$

we obtain from (2.13)-(2.15)

$$\varrho^* \mathbf{v}_t - \mu \Delta_y \mathbf{v} - \left(\frac{\mu}{3} + \zeta\right) \nabla_y \mathrm{div}\,_y \mathbf{v} + \gamma_2 \nabla_y \sigma = \mathbf{F}_2(\sigma, \mathbf{v}) \tag{5.4}$$

ME:lin2

$$\sigma_t + \varrho^* \mathrm{div}\,_y \mathbf{v} = G_2(\sigma, \mathbf{v}), \tag{5.5}$$

CE:lin2

$$\mathbf{v}|_{t=0} = \mathbf{u}_0 - \mathbf{V}(0), \quad \mathbf{v}|_{\partial\Omega_0} = 0. \tag{5.6}$$

icbc:lin

where $\gamma_2 = p'(\varrho^*)$ and

$$\mathbf{F}_2(\sigma, \mathbf{v}) = \mathbf{F}(\sigma + \varrho^*, \mathbf{v} + \mathbf{u}_{b2}) - \sigma \partial_t(\mathbf{v} + \mathbf{u}_{b2}) - [p'(\sigma + \varrho^*) - p'(\varrho^*)]\nabla_y \sigma \tag{5.7}$$

F:lin2

$$G_2(\sigma, \mathbf{v}) = G(\sigma + \varrho^*, \mathbf{v} + \mathbf{u}_{b2}) - (\sigma + \varrho^*)\mathrm{div}\,_y \mathbf{u}_{b2} - \sigma \mathrm{div}\,_y \mathbf{v} \tag{5.8}$$

G:lin2

5.2 Nonlinear estimates for the global well-posedness

We start with an analog of Lemma 4.2 which will be used to estimate the nonlinearities for large times.

nonlin4 **Lemma 5.2** *Let* $e^{\gamma t}(\sigma, \mathbf{v}) \in \mathcal{Y}(\infty)$ *for some* $\gamma > 0$ *and let* $\varrho_0, \mathbf{u}_0$ *satisfy the assumptions of Theorem 1.2. Then*

$$\|\mathbf{E}^0(\mathbf{k}_{\tilde{\mathbf{u}}}), \nabla_{\mathbf{k}_{\tilde{\mathbf{u}}}} \mathbf{E}^0(\mathbf{k}_{\tilde{\mathbf{u}}}),\|_{L_\infty((0,\infty)\times\Omega_0)} \leq C\|e^{\gamma t}(\sigma, \mathbf{v})\|_{\mathcal{Y}(\infty)}, \tag{5.9}$$

est:05

$$\sup_{t\in(0,\infty)}\|\sigma(\cdot,t)\|_{W^1_q(\Omega_0)} \leq C[\epsilon + \|e^{\gamma t}(\sigma, \mathbf{v})\|_{\mathcal{Y}(\infty)}], \tag{5.10}$$

est:06

$$\sup_{t\in(0,\infty)}\|\mathbf{v}(\cdot,t) - \mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}} \leq C\|e^{\gamma t}(\sigma, \mathbf{v})\|_{\mathcal{Y}(\infty)}, \tag{5.11}$$

est:07

$$\|\mathbf{v}\|_{L_\infty(0,\infty,W^1_\infty(\Omega_0))} \leq C\|e^{\gamma t}(\sigma, \mathbf{v})\|_{\mathcal{Y}(\infty)}, \tag{5.12}$$

est:08

where $\mathbf{k}_{\tilde{\mathbf{u}}}$ *is defined in* (2.10).

Proof: We have

$$\int_0^\infty \|\nabla_y \mathbf{v}\|_\infty \mathrm{d}t \le \left(\int_0^\infty e^{-\gamma t p'}\mathrm{d}t\right)^{1/p'} \left(\int_0^\infty e^{\gamma t p}\|\mathbf{v}\|_{W^2_q(\Omega_0)}\mathrm{d}t\right)^{1/p},$$

which implies (5.9). Next,

$$\|\sigma(\cdot,t)\|_{L_\infty(\Omega_0)} \le \|\varrho_0-\varrho^*\|_{L_\infty(\Omega_0)} + \int_0^t \|\sigma_t(s,\cdot)\|_{L_\infty(\Omega_0)}\mathrm{d}t$$

$$\le \epsilon + C\left(\int_0^t e^{-\gamma s p'}\mathrm{d}s\right)^{1/p'} \left(\int_0^\infty e^{\gamma s p}\|\tilde{\varrho}_t\|_{W^1_q(\Omega_0)}\mathrm{d}s\right)^{1/p},$$

which yields (5.10). Finally, (5.11) follows from Lemma 3.2, and (5.11) combined with Corollary 3.1 gives (5.12).

□

The following lemma gives estimates for the right hand sides of (5.4)-(5.5).

nonlin5 **Lemma 5.3** *Let* $\mathbf{F}_2(\tilde{\varrho},\tilde{\mathbf{u}}), G_2(\tilde{\varrho},\tilde{\mathbf{u}})$ *be defined in* (5.7) *and* (5.8). *Assume that* $\varrho_0, \mathbf{u}_0$ *and* $\mathbf{V}$ *satisfy* (1.13). *Then*

$$\|\mathbf{F}_2(\tilde{\varrho},\tilde{\mathbf{u}})\|_{L_p(0,\infty;L_q(\Omega_0))} + \|G_2(\tilde{\varrho},\tilde{\mathbf{u}})\|_{L_p(0,\infty;W^1_q(\Omega_0))} \le C(\|e^{\gamma t}(\sigma,\mathbf{v})\|^2_{\mathcal{Y}(\infty)} + \epsilon). \tag{5.13}$$ est:F

Proof. First, analogously to (4.17), this time using (5.9) and (5.2) we obtain

$$\begin{aligned}
&\|e^{\gamma t}A_{2\Delta}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla^2_y\tilde{\mathbf{u}},\ A_{1\Delta}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y\tilde{\mathbf{u}},\ A_{2\mathrm{div}\,,i}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla^2_y\tilde{\mathbf{u}},\ A_{1\mathrm{div}\,,i}(\mathbf{k}_{\tilde{\mathbf{u}}})\nabla_y\tilde{\mathbf{u}}\|_{L_p(0,\infty;L_q(\Omega_0))}\\
&\le C\|e^{\gamma t}(\sigma,\mathbf{v})\|_{\mathcal{Y}(\infty)}\|e^{\gamma t}(\mathbf{v}+\mathbf{u}_{b2})\|_{L_p(0,\infty;L_q(\Omega_0))}\\
&\le C\|e^{\gamma t}(\sigma,\mathbf{v})\|_{\mathcal{Y}(\infty)}[\|e^{\gamma t}(\sigma,\mathbf{v})\|_{\mathcal{Y}(\infty)} + \|e^{\gamma t}(\partial_t\tilde{\mathbf{V}},\nabla^2_y\tilde{\mathbf{V}})\|_{L_p(0,\infty;L_q(\Omega_0))}]\\
&\le C\|e^{\gamma t}(\sigma,\mathbf{v})\|_{\mathcal{Y}(\infty)}[\|e^{\gamma t}(\sigma,\mathbf{v})\|_{\mathcal{Y}(\infty)} + \epsilon].
\end{aligned}\tag{5.14}$$ est:A2

Next, by (5.10) and (5.2)

$$\|e^{\gamma t}\partial_t(\mathbf{v}+\mathbf{u}_{b2})\|_{L_p(0,\infty;L_q(\Omega_0))} \le \|\sigma\|_{L_\infty((0,\infty)\times\Omega_0)}\|e^{\gamma t}\partial_t(\mathbf{v}+\mathbf{u}_{b2})\|_{L_p(0,\infty;L_q(\Omega_0))} \le C[\varepsilon + \|e^{\gamma t}(\sigma,\mathbf{v})\|_{\mathcal{Y}(\infty)}]^2,$$

and

$$\|e^{\gamma t}[p'(\tilde{\varrho})-p'(\varrho^*)]\nabla_y\sigma\|_{L_p(0,\infty;L_q(\Omega_0))} \le \|\sigma\|_{L_\infty((0,\infty)\times\Omega_0)}\|e^{\gamma t}\nabla_y\sigma\|_{L_p(0,\infty;L_q(\Omega_0))} \le C[\varepsilon + \|e^{\gamma t}(\sigma,\mathbf{v})\|_{\mathcal{Y}(\infty)}]^2.$$

Combining all above estimates we get the required estimate for $\|\mathbf{F}_2\|_{L_p(0,\infty;L_q(\Omega_0))}$. Finally, G_2 and its space derivatives are estimated in a similar way using Lemma 5.2 and (5.2).

□

5.3 Proof of Theorem 1.2

It is now easy to verify the following estimate which allows to prolong the local solution for arbitrarily large times.

Lemma 5.4 *Assume* $\sigma, \mathbf{v}$ *is solution to* (5.4)-(5.6) *with* $\varrho_0, \mathbf{u}_0$ *and* $\mathbf{V}$ *satisfying the assumptions of Theorem 1.2. Then*

$$\|e^{\gamma t}(\sigma,\mathbf{v})\|_{\mathcal{Y}(\infty)} \le E(\epsilon). \tag{5.15}$$ est:expn

Proof. Combining Proposition 3.2 and Lemma 5.3 we obtain

$$\|e^{\gamma t}(\sigma, \mathbf{v})\|_{\mathcal{Y}(T)} \leq C[\epsilon + \|e^{\gamma t}(\sigma, \mathbf{v})\|^2_{\mathcal{Y}(T)}]. \tag{5.16}$$

est:expno

Note that we derived this inequality for $T = \infty$, however it is easy to observe that the same arguments yield (5.16) for any $T > 0$. Consider the equation

$$x^2 - \frac{x}{C} + \epsilon = 0.$$

Its roots are

$$x_1(\epsilon) = \frac{1}{2C} - \sqrt{\frac{1}{4C^2} - \epsilon}, \quad x_2(\epsilon) = \frac{1}{2C} + \sqrt{\frac{1}{4C^2} - \epsilon}.$$

Notice that the inequality (5.16) implies either $\|e^{\gamma t}(\sigma, \mathbf{v})\|_{\mathcal{Y}(T)} \leq x_1(\epsilon)$ or $\|e^{\gamma t}(\sigma, \mathbf{v})\|_{\mathcal{Y}(T)} \geq x_2(\epsilon)$. However,

$$\|e^{\gamma t}(\sigma, \mathbf{v})\|_{\mathcal{Y}(T)} \to 0$$

as $T \to 0$, therefore

$$\|e^{\gamma t}(\sigma, \mathbf{v})\|_{\mathcal{Y}(T)} \leq x_1(\epsilon)$$

for T small. Finally, $\|e^{\gamma t}(\sigma, \mathbf{v})\|_{\mathcal{Y}(T)}$ is continuous in time and therefore

$$\|e^{\gamma t}(\sigma, \mathbf{v})\|_{\mathcal{Y}(\infty)} \leq x_1(\epsilon).$$

□

Now it is a standard matter to prolong the local solution for arbitrarily large times. For this purpose it is enough to observe that if the initial data satisfies the smallness assumption from Theorem 1.2 then the time of existence from Theorem 1.1 satisfies $T > C(\epsilon) > 0$. Therefore, for arbitrarily large T^* we can obtain a solution on $(0, T^*)$ in a finite number of steps. By the estimate (5.15) this solution satisfies (1.14)-(1.15).

Finally, the equivalence of norms can be justified as in Section 4.4, using (5.12) instead of (4.13).

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