Exercises for Mathematical Logic (26 Oct 2022)

11. (If you are familiar with topology.) Give a direct proof of the propositional compactness theorem, not using the completeness theorem.
[Hint: Consider the product topology on the set $\{0,1\}^A$ of all assignments.]

12. Prove that if a term $t(x_0,\dots,x_{n-1})$ is free for y in a formula $\varphi(x_0,\dots,x_{n-1},y)$, then for all terms $s_0,\dots,s_{n-1}$, the formula $\big(\varphi(t/y)\big)(s_0/x_0,\dots,s_{n-1}/x_{n-1})$ is syntactically identical to the formula $\varphi\big(s_0/x_0,\dots,s_{n-1}/x_{n-1},t(s_0/x_0,\dots,s_{n-1}/x_{n-1})/y\big)$.

13. A set of propositional or first-order sentences S is *independent* if S is not equivalent to S' for any proper subset $S' \subsetneq S$.

(i) S is independent iff $S \smallsetminus \{\varphi\} \nvDash \varphi$ for all $\varphi \in S$.

(ii) Show that every countable theory T has an independent axiomatization, i.e., an independent set of sentences S equivalent to T. [Hint: Try to generalize the fact that $\{\varphi,\psi\} \equiv \{\varphi,\psi \vee \neg\varphi\}$.]

(Uncountable theories have independent axiomatizations as well by a theorem of I. Reznikoff, but this is more difficult to prove.)

14. Consider a modification of the first-order proof system given in the lecture such that the axioms of equality are replaced with the axiom $x = x$ and the axiom schema $t = s \wedge \varphi(t/s) \to \varphi(s/x)$ for all formulas φ and terms t, s free for x in φ. Show that this is equivalent to the original proof system.